B. Math. III and M. Math. I Topology Mid-Semestral Examination 2011

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Each question carries 10 marks. Attempt all questions. Anything proved in the class maybe cited without proof. Results of exercises, however, must be derived in full. You may use books and notes.

(1) (i): Let X be a well-ordered set. Show that every non-empty subset A of X has a greatest lower bound.

(ii): Give an example of an element $x \in S_\Omega$ (the first uncountable ordinal space) such that the singleton set $\{x\}$ has no *strict* greatest lower bound. (i.e. x has no immediate predecessor). Justify your answer.

(2) (i): Let A and B be subsets of a topological space X. Show that

$$(\mathrm{Int}A) \ \cup \ (\mathrm{Int}B) \subset \mathrm{Int}(A \cup B)$$

where, for a subset A of X, $\mathrm{Int}A$ denotes the interior of A, defined by

$$\mathrm{Int}A = \bigcup \{V : V \text{ is open; } V \subset A\}$$

(ii): Give an example to show that the inclusion in part (i) above may be strict.

(3) (i): Let $\mathbb{R}$ be given its usual order, and let $\mathbb{R} \times \mathbb{R}$ be given the dictionary order. Prove that the order topology from this dictionary order is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where the first factor $\mathbb{R}_d$ is the set of reals with the discrete topology, and the second factor is $\mathbb{R}$ with its usual topology.

(ii): Let X be a connected metric space containing at least two points. Show that X is uncountable.

(4) (i): Show that there does not exist a continuous surjection $f : \mathbb{R} \to S_\Omega$, where $\mathbb{R}$ is the space of reals with usual topology and S_Ω is as in Q. 1 above.

(ii): Let $K := \{1/n \ : \ n \in \mathbb{N}\}$ and let $\mathbb{R}_K$ denote the topological space given by the set of reals with the K-topology, which is generated by the basis :

$$\mathcal{B} = \{(a, b) : \ a, b \in \mathbb{R}\} \cup \{(a, b) \setminus K : \ a, b \in \mathbb{R}\}$$

Show that $[0, 1]$ is not a compact subset of $\mathbb{R}_K$.